

A bioeconomic model of a single-species fishery with a marine reserve

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Abstract

This study examines the impact of the creation of marine protected areas (MPAs), from both economic and biological perspectives. In particular, we examine the effects of protected patches and harvesting on resource populations. We conclude that protected patches are an effective means of conserving resource populations, even though extinction cannot be prevented in all cases. We discuss the dynamic optimization of a harvest policy by choosing $E(t)$, the harvesting effort, as the dynamic variable. We also discuss the optimal equilibrium harvest policy and explain the biological and bioeconomic interpretations of the results.

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1. Introduction

Marine protected areas (MPAs), which we define as spatially well-defined areas where no harvesting takes place, have become a popular approach to managing marine fisheries (Polacheck, 1990; Holland and Brazee, 1996; Sanchirico and Wilen, 1999, 2001, 2005; Bonocoeur et al., 2002; Anderson, 2002; Luck et al., 1998; Rodwell et al., 2002; Neubert, 2003; Hilborn et al., 2006). The impetus for establishing MPAs originally came from countries such as Australia, New Zealand, and the Seychelles, where MPAs have provided many general benefits as a tool for conservation and marine environmental management. It is relatively easy to enforce fishing bans in MPAs, if the necessary will and resources exist: quite simply, any fisherman working in such a reserve is breaking the rules. The more effectively a reserve is functioning, the more carefully these restrictions must be enforced.

Two core objectives have motivated the establishment of most marine reserves: conservation and sustainable provision for human use. The conservation goals include: (i) the

conservation of biodiversity; (ii) the conservation of rare and restricted-range species; (iii) the maintenance of genetic diversity; (iv) the maintenance and/or restoration of the natural ecosystem on both local and regional scales; and (v) the conservation of areas vital for vulnerable life stages. The goals relating to human use include: (i) the management of fisheries, (ii) recreation, (iii) education, (iv) research, and (v) the fulfillment of esthetic needs. The current overall enthusiasm for reserves and the recent rapid increase in new policies promoting reserve formation have motivated this paper.

Not everyone supports a major expansion of marine reserves, of course, and fishermen are among the most vocal skeptics. One issue for fishermen is whether they will incur significant costs in the form of lost access to traditional grounds. Whether reserves are likely to be costly to fishermen is a complicated issue. We know, for example, that closing all coastal habitats would essentially cost fishermen all of their current income, and we also know that doing nothing would leave us with the status quo in fisheries management. Several different arguments have been put forth suggesting that reserves at this level would actually benefit fishermen. One argument is that current management methods are not sufficiently strict and are destined to fail over the long run (Ludwig et al., 1993). More interesting arguments have suggested that closing some areas might actually enhance fisheries productivity.

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This argument is based on the possibility that the existence of larger protected populations of more fecund individuals within reserve boundaries might actually increase the flow of larva outside the boundaries, thus enhancing fishery output in the remaining open areas. MPAs also provide important sites for study and experimentation, including control sites for studies of fishing effects. These have been widely used for education, tourism, and increasing the public awareness and understanding of environmental issues. The net economic benefits from some of these activities can far exceed those from fisheries.

Despite the growing interest in marine reserves (Suman et al., 1999; Gell and Roberts, 2002; Halpern, 2003; Guenette et al., 1998; Brown and Roughgarden, 1997), both the economic benefits and the conservation impact of marine reserves have recently been questioned (Hannesson, 1998; Allison et al., 1998). Conrad (1999) showed that, in the absence of ecological uncertainty and in the context of optimal harvesting, reserves generate no economic benefits to fishermen. Such a result coincides with the perspectives of many fishermen and also some economists. However, Luck et al. (1998) asserted that MPAs can be viewed as a kind of insurance against scientific uncertainty, stock assessments, or regulation errors.

Following the tradition of Clark (1985), we employ a simple modeling approach in this paper. That is, we endeavor to understand complex management issues by developing and analyzing a simple model. Despite the simplifying assumptions used, we still manage to include in the analysis several major biological and management characteristics, including population growth, migration, harvesting effort, and reserve size. To derive analytic results concerning the effects on population level and yield of the division of a population’s habitat area into sub-areas, namely a nature reserve (NR) and a harvesting zone (HZ), we construct and analyze a simple logistic growth model. This modeling approach allows us to characterize explicitly the relationships among population protection, yield, reserve size, optimal harvest efforts, and biological parameters, including population growth and migration.

2. Model

2.1. Pre-reserve

We assume that the pre-reserve population is uniformly distributed across its habitat area, with no distinction between patches for mating, spawning, growth, maturation, or protection, for example. We also assume that the population evolves according to the logistic law of growth, which is described by

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right), \tag{1}$$

where x is the population size at any time t , k the carrying capacity, and r the intrinsic growth rate. Now suppose that

the population described by Eq. (1) is subjected to harvesting. We use the catch-per-unit-effort (CPUE) hypothesis (Clark, 1990) to refer to the assumption that the harvest is proportional to the stock level, or that qEx , where E and q are positive constants denoting the harvesting effort and catchability coefficients, respectively.

The equation characterizing the harvested population becomes

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - qEx.$$

If $E > r/q$, the right-hand side is always negative, and a rapid collapse of the resource population will occur. In this case, extinction of the resource population is inevitable, and the ecological environment is destroyed.

2.2. Post-reserve

To protect the population and ecological environment, the region is divided into two patches. The two subpopulations are assumed to be homogeneously distributed across their sub-areas. In our model, we assume that each subpopulation has its own carrying capacity, which is proportionate to its distribution area. We also assume that the total population distribution area equals unity, and that sub-areas 1 (NR) and 2 (HZ) equal s and $1-s$, respectively, where $0 < s < 1$. If $s = 0$, then there is no reserve, and this possibility provides the status quo reference case (for more details, see Anderson (2002)). We assume that fish are free to migrate between the two patches, as well as within each one of them. The net migration between the two areas depends on the difference in the subpopulation densities. We also assume that each subpopulation has its own carrying capacity, which is proportionate to its distribution area.

Thus, our model becomes

$$\begin{aligned} \frac{dx_1}{dt} &= rx_1\left(1 - \frac{x_1}{sk}\right) - \sigma\left(\frac{x_1}{sk} - \frac{x_2}{(1-s)k}\right), \\ \frac{dx_2}{dt} &= rx_2\left(1 - \frac{x_2}{(1-s)k}\right) + \sigma\left(\frac{x_1}{sk} - \frac{x_2}{(1-s)k}\right). \end{aligned} \tag{2}$$

Normalizing the population by dividing the population level by the carrying capacity and considering the harvest in HZ based on a CPUE that is proportional to the population density, we arrive at the growth equations

$$\begin{aligned} \frac{dx_1}{dt} &= rx_1\left(1 - \frac{x_1}{s}\right) - \sigma\left(\frac{x_1}{s} - \frac{x_2}{1-s}\right) = f(x_1, x_2), \\ \frac{dx_2}{dt} &= rx_2\left(1 - \frac{x_2}{(1-s)}\right) + \sigma\left(\frac{x_1}{s} - \frac{x_2}{1-s}\right) - qEx_2 \\ &= g(x_1, x_2), \end{aligned} \tag{3}$$

where σ is the migration coefficient.

3. Behavior under harvesting

The behavior of a nonlinear system (3) can be characterized by standard equilibrium, stability, and bifurcation analyses.

In the following theorem, we show that all the solutions of the model under study are bounded.

Theorem 1. All the solutions of system (3) that initiate in R_2^+ are uniformly bounded.

Proof. We define the function

$$W = x_1 + x_2. \tag{4}$$

The time derivative of W along a solution of system (3) is

$$\dot{W} = rx_1 \left(1 - \frac{x_1}{s}\right) + rx_2 \left(1 - \frac{x_2}{(1-s)}\right) - qEx_2.$$

For each $\tau > 0$, we have that

$$\begin{aligned} \dot{W} + \tau W &= rx_1 \left(1 - \frac{x_1}{s}\right) + rx_2 \left(1 - \frac{x_2}{(1-s)}\right) \\ &\quad - qEx_2 + \tau x_1 + \tau x_2 \\ &\leq \frac{rs}{4} \left(1 + \frac{\tau}{r}\right)^2 + \frac{r(1-s)}{4} \left(1 - \frac{qE}{r} + \frac{\tau}{r}\right)^2. \end{aligned} \tag{5}$$

Thus, there exists a $\mu > 0$ with $\dot{W} + \tau W < \mu$. Applying the theorem on differential inequality (Birkhoff and Rota, 1982), we obtain

$$0 \leq W(x_1, x_2) \leq (\mu/\tau)(1 - \exp(-\tau t)) + W(x_1(0), x_2(0)) \exp(-\tau t).$$

And, for $t \rightarrow \infty$, we have $0 \leq W \leq \mu/\tau$.

Therefore, we have

$$B = [(x_1, x_2) \in R_2^+ : W < \mu/\tau + \varepsilon, \text{ for any } \varepsilon > 0],$$

where B is the region in which all the solutions of system (3) that start in R_2^+ are confined. \square

3.1. Equilibrium and stability

We now check whether the model has an equilibrium point with positive subpopulations, as well as whether this equilibrium is stable. The equilibrium for each of the subpopulations requires $(dx_1/dt) = (dx_2/dt) = 0$, and this obviously implies $dx/dt = 0$.

From Eq. (3), we get

$$x_2 = \phi(x_1) = Ax_1(x_1 - B), \tag{6}$$

$$x_1 = \varphi(x_2) = Cx_2(x_2 - D), \tag{7}$$

where the parameters are defined as

$$A = \frac{(1-s)r}{s\sigma}, \quad B = s - \frac{\sigma}{r}, \quad c = \frac{sr}{(1-s)\sigma},$$

$$D = 1 - s - \frac{\sigma}{r} - \frac{qE(1-s)}{r}.$$

Explicit biological interpretations of the parameters A , B , C , and D are not available, but these conditions must be

satisfied by the parameters if the equilibrium point is to exist.

Here $A > 0$ and $C > 0$. Curve (6), which expresses the isocline $dx_1/dt = 0$, we refer to as S_1 , and we refer to curve (7), which expresses the isocline $dx_2/dt = 0$, as S_2 .

Both of these curves are parabolas. The axis of curve S_1 is parallel to the x_2 -axis, and the axis of curve S_2 is parallel to the x_1 -axis. If $B > 0$, curve S_1 enters the positive quadrant at $(B, 0)$, and, if $B < 0$, then it enters through the origin. Similarly, if $D > 0$, the curve S_2 enters the positive quadrant at $(0, D)$, and, if $D < 0$, then it enters the positive quadrant through the origin. All of these possibilities are illustrated in Figs. 1–5.

Given these possible cases, it is clear that there is a chance of extinction only when the parameters B and D are both negative (see Fig. 2). Hence, the question of how to avoid extinction by harvesting remains to be answered. With reference to Fig. 1, we consider the slopes of S_1 and S_2 at the origin. If

$$\left. \frac{dx_2}{dx_1} \right|_{S_1}(0) < \left. \frac{dx_2}{dx_1} \right|_{S_2}(0), \tag{8}$$

then there exists a positive equilibrium (x_1^*, x_2^*) , while, for the converse of inequality (8), there is no positive equilibrium (Fig. 2).

For the existence of a positive equilibrium, we require, from (6), (7), and (8), that

$$E < E_c = \frac{r}{q} - \frac{s\sigma}{q(s - \sigma/r)(1-s)} > \frac{r}{q}, \tag{9}$$

since $B = s - (\sigma/r) < 0$.

Differentiating E_c with respect to s , we observe that, for a fixed $\sigma \in (0, r)$, E_c increases for $s \in (0, \sqrt{\sigma/r})$, decreases for $s \in (\sqrt{\sigma/r}, 1)$, and attains its maximum at $s = \sqrt{\sigma/r}$. When $\sigma \geq r$, E_c increases with s . On the other hand, differentiating E_c with respect to σ , we observe that, for fixed $s \in (0, 1)$, E_c decreases with σ .

The dynamic behavior of the equilibria can be studied by computing the variational matrix corresponding to each equilibrium. In particular, the equilibrium $(0, 0)$ is

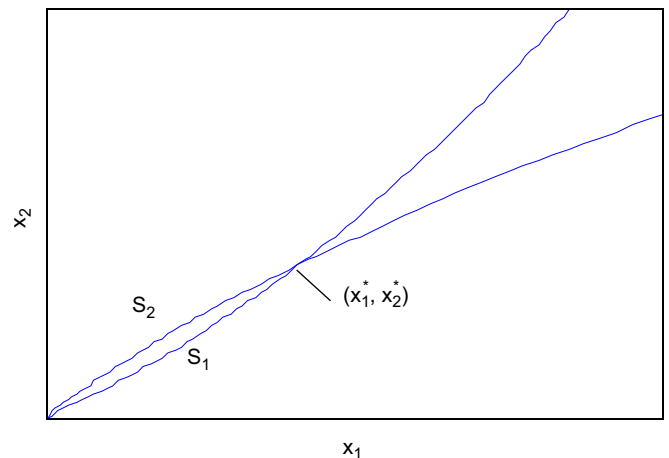


Fig. 1. Case of $B < 0, D < 0$, where a positive equilibrium exists.

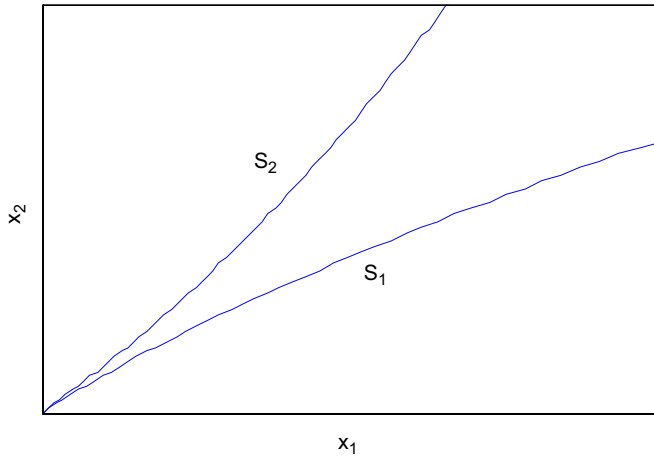


Fig. 2. Case of $B < 0, D < 0$, where no positive equilibrium exists.

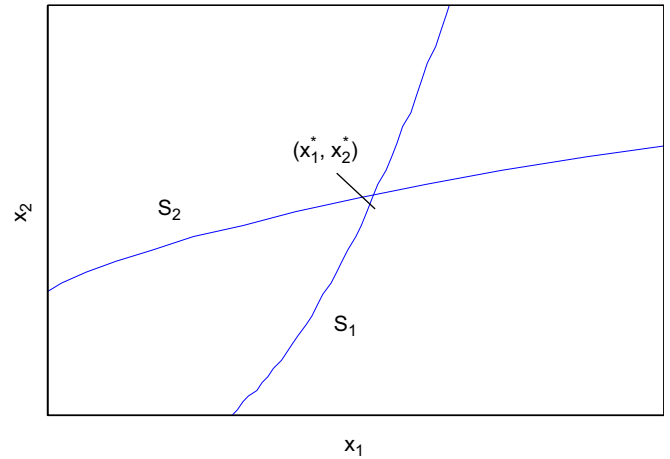


Fig. 5. Case of $B > 0, D > 0$.

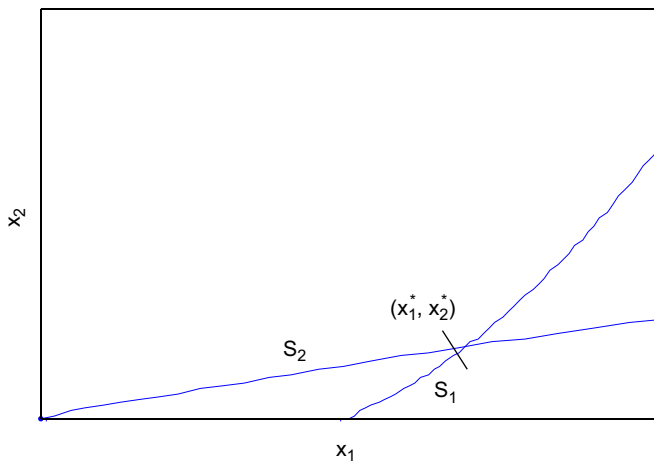


Fig. 3. Case of $B > 0, D < 0$.

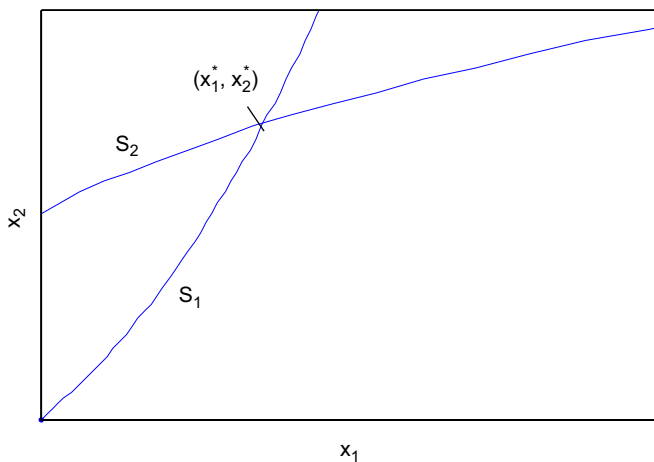


Fig. 4. Case of $B < 0, D > 0$.

real parts, and, hence, that it is locally asymptotically stable. In addition, the interior equilibrium point is globally asymptotically stable (see appendix).

From system (3), we see that the sustainable yield for the model is given by

$$Y = rx_1 \left(1 - \frac{x_1}{s}\right) + rx_2 \left(1 - \frac{x_2}{1-s}\right) \tag{10}$$

which indicates that migration does not matter for the sustainable yield level, even though the relative distribution of the population in the two sub-areas does matter.

4. The effects of a protected patch on the population

From Fig. 2, it is clear that, if $E > (r/q) - [s\sigma/q(s - (\sigma/r))(1 - s)]$, the introduction of a protected patch is not sufficient to prevent extinction. Therefore, we can conclude that, if E increases in $(0, (r/q) - [s\sigma/q(s - (\sigma/r))(1 - s)])$, the interior equilibrium (x_1^*, x_2^*) decreases monotonically and, ultimately, that $(x_1^*, x_2^*) \rightarrow (0, 0)$. So, if the harvesting effort is large ($E > (r/q) - [s\sigma/q(s - (\sigma/r))(1 - s)]$), the population becomes extinct. In the absence of any protected patches, the population inevitably becomes extinct as long as $E \geq (r/q)$. In this case, a protected patch is still viable, even though it does not prevent extinction in all situations, because it prevents extinction for

$$\frac{r}{q} < E < \frac{r}{q} - \frac{s\sigma}{q(s - \frac{\sigma}{r})(1 - s)}.$$

Figs. 6 and 7 clearly indicate that larger reserves always increase biomass; however, a larger reserve does not necessarily increase the harvest. This result is obtained because, although a larger reserve increases spillovers to the fishery, it also reduces the proportion of the population that is subject to harvesting. Putting these pieces together, we can draw the final conclusion that the maintenance of a protected region is practicable in the management of resource populations, although in some cases extinction cannot be prevented.

unstable. Using the Routh–Hurwitz criteria, it is easy to check that all eigenvalues of the variational matrix corresponding to the interior equilibrium have negative

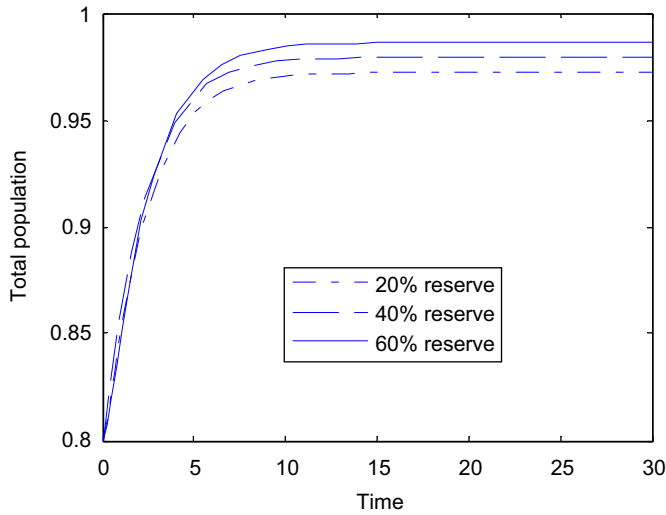


Fig. 6. The relationship between total population and reserve size. The parameter values are $r = 0.5$, $q = 0.1$, $\sigma = 0.4$, and $E = 0.1738$.

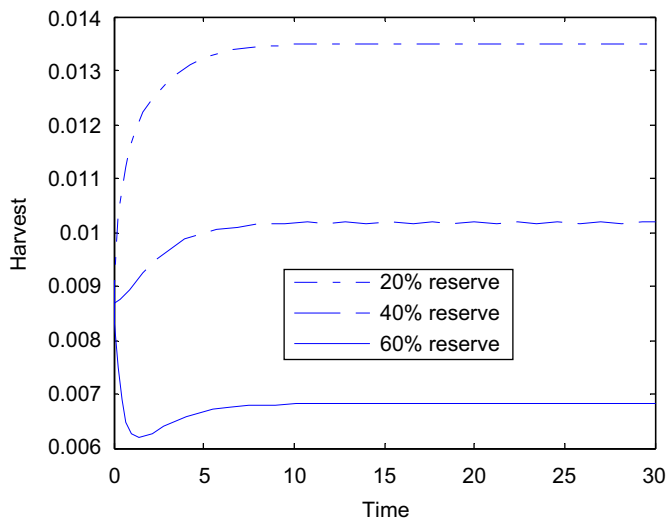


Fig. 7. Harvest against time. The parameter values are the same as in Fig. 6.

In practice, environmental conservation should be valued more highly than the economic benefits of resource management, since it is generally unwise to eliminate any resource population. The ultimate purpose of establishing protected patches for resource populations is to produce greater overall benefits.

5. Optimal harvesting: dynamic optimization

5.1. Problem formulation and necessary conditions

Suppose that a governing body manages the fishery by setting up a reserve zone to control the harvesting of fish. Its objective is to exploit the available resources optimally. The present value J of a continuous time-stream of

revenues is given by

$$J = \max_E \int_0^\infty e^{-\delta t} (pqx_2(t) - c(x_2))E(t) dt, \tag{11}$$

where δ is the instantaneous rate of annual discount, p the price of a unit of fish, and $c(x_2)$ the cost of harvesting. The cost of fishing is assumed to decrease with the stock size, i.e., $\partial c(x_2)/\partial x_2 \leq 0$. Thus, our objective is to maximize J subject to the state Eqs. (3) and the control constraint that

$$0 \leq E \leq E_{\max} < E_c. \tag{12}$$

Applying Pontryagin’s maximum principle (Kamien and Schwartz, 1981), we find that the current value Hamiltonian is given by

$$H = (pqx_2(t) - c(x_2))E(t) + \lambda_1 f(x_1, x_2) + \lambda_2 g(x_1, x_2). \tag{13}$$

The equations that form the necessary conditions for a solution are as follows:

- condition for a maximum:

$$\left(\max_E \right) H \Rightarrow \left(\max_E \right) (pqx_2(t) - c(x_2) - \lambda_2 qx_2)E(t), \tag{14}$$

$$0 \leq E \leq E_{\max},$$

where E_{\max} is the maximum harvest rate.

- equations of motion for the co-state or shadow prices:

$$\begin{aligned} \dot{\lambda}_1 &= \delta \lambda_1 - \frac{\partial H}{\partial x_1} = \delta \lambda_1 - \lambda_1 \frac{\partial f}{\partial x_1} - \lambda_2 \frac{\partial g}{\partial x_1} \\ &= \delta \lambda_1 - \lambda_1 \left(r - \frac{2rx_1}{s} - \frac{\sigma}{s} \right) - \lambda_2 \frac{\sigma}{s}, \end{aligned} \tag{15}$$

$$\begin{aligned} \dot{\lambda}_2 &= \delta \lambda_2 - \frac{\partial H}{\partial x_2} = \delta \lambda_2 - pqE + \frac{\partial c}{\partial x_2} E - \lambda_1 \frac{\partial f}{\partial x_2} - \lambda_2 \frac{\partial g}{\partial x_2} \\ &= \delta \lambda_2 - pqE + \frac{\partial c}{\partial x_2} E - \lambda_1 \frac{\sigma}{1-s} \\ &\quad - \lambda_2 \left(r - \frac{2rx_2}{1-s} - \frac{\sigma}{1-s} - qE \right), \end{aligned} \tag{16}$$

- transversality conditions, since $x_1 \geq 0$ and $x_2 \geq 0$:

$$\lim_{t \rightarrow \infty} \lambda_1 \geq 0, \quad \lim_{t \rightarrow \infty} \lambda_2 \geq 0.$$

Given the linear form of the harvesting cost function, $c(x_2)E$, the Hamiltonian (13) depends linearly on E with coefficient $(pqx_2 - (c(x_2) - \lambda_2 qx_2))$. Consequently, its maximum value is reached for the extremes of E ; i.e., the harvest rate must be either 0 or E_{\max} . This observation leads to the rule that one must fish as much as possible when the shadow price of fish is sufficiently low ($\lambda_2 < p - (c(x_2)/qx_2)$), and not fish at all when the shadow price is sufficiently high ($\lambda_2 > p - (c(x_2)/qx_2)$). Furthermore, when $\lambda_2 = p - (c(x_2)/qx_2)$, the harvest rate is undetermined. In this case, three solutions for E are

possible, namely 0, E_{\max} , or \tilde{E} , which is the singular control that maintains the condition $\lambda_2 = p - (c(x_2)/qx_2)$. Therefore, the optimal control path will be either “bang–bang” (i.e., harvesting maximally, not harvesting at all, or alternating between the two) or singular (i.e., equating revenues with the shadow price).

We assume that there exists a unique optimal path. After an initial period, the optimal trajectory of the system will approach either an equilibrium or a cycle. Suppose that the optimal path does not approach an equilibrium; in this case, it must cross itself. At this point, the optimal path must continue as it did before, or it will not be unique. This implies a cycle. So, finally, the optimal trajectory will reach either an equilibrium or a cycle. We refer to this state as the end state. The approach path is the beginning of the optimal trajectory prior to the end state being reached.

We devote the remainder of this section to examining the end state. Remember that the harvest rate of an optimal trajectory, and thus of the end state, must be “bang–bang,” singular, or a combination of the two. In consequence, the following four end states are conceivable:

- (1) *No harvesting*: $E = 0$ and $\lambda_2 \geq p - (c(x_2)/qx_2)$. The end state is an equilibrium, and the harvest rate is part of a “bang–bang” control (see Section 5).
- (2) *Maximum harvesting*: $E = E_{\max}$ and $\lambda_2 \leq p - (c(x_2)/qx_2)$. The end state is an equilibrium, and the harvest rate is part of a “bang–bang” control (see Section 5).
- (3) *A singular state*: $E = \tilde{E}$ and $\lambda_2 = p - (c(x_2)/qx_2)$ (see Section 5). A singular harvest rate is applied. This can result in two types of singular equilibria (see Section 5).
- (4) A “bang–bang” cycle, which is an oscillation that is controlled by a harvest rate that alternates among the maximum, zero, and possibly a singular harvest rate:

$$E = E_{\max} \quad \text{when } \lambda_2 < p - \frac{c(x_2)}{qx_2},$$

$$E = 0 \quad \text{when } \lambda_2 > p - \frac{c(x_2)}{qx_2},$$

$$E = \tilde{E} \quad \text{when } \lambda_2 < p - (c(x_2)/qx_2).$$

No harvesting (Case 1): The first possibility is straightforward. Not harvesting is optimal when, at the equilibrium (x_1^*, x_2^*) , the total cost of fishing $c(x_2) + \lambda_2$ exceeds the price of fish. This means that, at any harvest rate, the loss in the social value of fish exceeds the net gain from the fishery.

Maximum harvesting (Case 2): The second possibility is to continue harvesting at the maximum level. In this case, the price of fish must exceed the total costs. If the maximum harvest level is relatively small ($E_{\max} < E_c$), then the system asymptotically approaches the equilibrium (x_1^*, x_2^*) .

A singular case: The third possibility is an end state in which the total system (state and co-state) remains in a

singular state. From condition (14) it follows that, at this end state, we have to satisfy

$$\lambda_2 = p - \frac{c(x_2)}{qx_2}. \tag{17}$$

Substituting (17) and its derivative, $\dot{\lambda}_2 = -(1/q)[(x_2(\partial c/\partial x_2) - c(x_2))/(x_2^2)]g(x_1, x_2)$, into (16) yields the following expression:

$$\lambda_1 = \left[\delta x_2 - \left(pq - \frac{\partial c}{\partial x_2} \right) E - \left(p - \frac{c}{qx_2} \right) \frac{\partial g}{\partial x_2} + \frac{g(x_1, x_2)x_2(\partial c/\partial x_2) - c(x_2)}{q x_2^2} \right] \bigg/ \frac{\partial f}{\partial x_2}. \tag{18}$$

We take the time derivative of expression (18) for λ_1 and substitute it, together with (17), into (15). This action eliminates both shadow prices, and we obtain the following expression, which implicitly defines \tilde{E} , the harvest rate in the singular state:

$$\begin{aligned} & \left[\delta q x_2^4 + x_2^4 \frac{\partial^2 c}{\partial x_2^2} q \tilde{E} - p q x_2^4 \frac{\partial^2 g}{\partial x_2^2} - c \frac{\partial g}{\partial x_2} x_2^2 \right. \\ & + c x_2^3 \frac{\partial^2 g}{\partial x_2^2} + x_2^3 \frac{\partial^2 c}{\partial x_2^2} g - 2 x_2^2 \frac{\partial c}{\partial x_2} g + 2 c x_2 g \\ & \left. + \left(x_2^3 \frac{\partial c}{\partial x_2} - x_2^2 c \right) \frac{\partial g}{\partial x_2} \right] g \frac{\partial f}{\partial x_2} \\ & - \left[\delta x_2 - \left(pq - \frac{\partial c}{\partial x_2} \right) \tilde{E} - \left(p - \frac{c}{qx_2} \right) \frac{\partial g}{\partial x_2} \right. \\ & \left. + \frac{g(x_1, x_2)x_2(\partial c/\partial x_2) - c(x_2)}{q x_2^2} \right] g q x_2^4 \frac{\partial^2 f}{\partial x_2^2} \\ & = q x_2^4 \left(\delta - \frac{\partial f}{\partial x_1} \right) \left[\delta x_2 - \left(pq - \frac{\partial c}{\partial x_2} \right) \right. \\ & \left. \tilde{E} - \left(p - \frac{c}{qx_2} \right) \frac{\partial g}{\partial x_2} + \frac{g(x_1, x_2)x_2(\partial c/\partial x_2) - c(x_2)}{q x_2^2} \right] \\ & \times \frac{\partial f}{\partial x_2} - q x_2^4 \frac{\partial g}{\partial x_1} \left(\frac{\partial f}{\partial x_2} \right)^2 \left(p - \frac{c}{qx_2} \right). \end{aligned} \tag{19}$$

We have ignored the function’s arguments so as not to further complicate the expression. Eq. (19) indicates that it is possible to identify a singular harvest rate for every point in the phase diagram. Be aware, however, that $0 < \tilde{E} < E_{\max}$, and, thus, it may not be feasible to find an \tilde{E} for every value of x_1 and x_2 .

If a singular harvest rate is employed, an autonomous system results that describes the singular trajectories:

$$\begin{aligned} \frac{dx_1}{dt} &= r x_1 \left(1 - \frac{x_1}{s} \right) - \sigma \left(\frac{x_1}{s} - \frac{x_2}{1-s} \right) \\ \frac{dx_2}{dt} &= r x_2 \left(1 - \frac{x_2}{1-s} \right) + \sigma \left(\frac{x_1}{s} - \frac{x_2}{1-s} \right) - q \tilde{E} x_2. \end{aligned} \tag{20}$$

Here \tilde{E} is the singular harvest rate, which is implicitly given by Eq. (19). From system (20), it follows that, in equilibrium, $\tilde{E} = (1/q)[r(1 - x_2^*/(1-s)) + (\sigma/x_2^*)(x_1^*/s - (x_2^*/(1-s)))]$, which is independent of time. Thus, the singular harvest rate is constant in equilibrium. Therefore, the equilibria of

the singular system must be equal to the equilibria under fixed harvest rates, as noted in Section 3. Semmler and Sieveking (1994) showed that constant optimal harvesting may push a predator–prey system into cyclical behavior, whereas such a system would reach equilibrium in the absence of harvesting. This result does not hold for the system studied here.

6. Optimal harvesting: equilibrium solutions

We will henceforth analyze a less general case. We take the cost of harvesting to be $c(x_2) = c$ (a constant). The Hamiltonian is given by Eq. (13). From the necessary conditions for optimality (Clark, 1990), the following condition is valid along the optimal solution:

$$\frac{\partial H}{\partial E} = pqx_2 - c - \lambda_2 qx_2 = 0, \tag{21}$$

where the dynamics of the associated shadow prices are described by Eqs. (15) and (16). We observe that, along an optimal equilibrium solution of the problem under consideration, we have

$$\delta\lambda_1 - \lambda_1 \left(r - \frac{2rx_1}{s} - \frac{\sigma}{s} \right) - \lambda_2 \frac{\sigma}{s} = 0, \tag{22}$$

$$\delta\lambda_2 - pqE - \lambda_1 \frac{\sigma}{1-s} - \lambda_2 \left(r - \frac{2rx_2}{1-s} - \frac{\sigma}{1-s} - qE \right) = 0. \tag{23}$$

In view of Eq. (21), if the equilibrium solution $(x_1^*(E), x_2^*(E))$ satisfies

$$\delta - r + \frac{2rx_1^*(E)}{s} + \frac{\sigma}{s} \neq 0,$$

then we have

$$\begin{aligned} & \left(\delta - r + \frac{2rx_1^*(E)}{s} + \frac{\sigma}{s} \right) \left[\left(p - \frac{c}{qx_2^*} \right) \right. \\ & \times \left. \left(\delta - r + \frac{2rx_2^*}{1-s} + \frac{\sigma}{1-s} - \frac{cE}{x_2} \right) \right] \\ & - \frac{\sigma^2}{s(1-s)} \left(p - \frac{c}{qx_2^*} \right) = 0. \end{aligned} \tag{24}$$

Alternatively, along the optimal solution, we have (from (22) and (23))

$$\lambda_1 = \frac{\sigma pqE}{sD}$$

and

$$\lambda_2 = \frac{pqE(\delta - r + (2rx_1^*(E)/s) + (\sigma/s))}{D},$$

where

$$\begin{aligned} D = & \left(\delta + \frac{rx_1^*(E)}{s} + \frac{\sigma}{1-s} \frac{x_2^*}{x_1^*} \right) \left(\delta + \frac{rx_2^*}{1-s} + \frac{\sigma}{s} \frac{x_1^*}{x_2^*} \right) \\ & - \frac{\sigma^2}{s(1-s)} > 0. \end{aligned}$$

At the optimal steady state, the optimal fishing effort can be derived as

$$E^* = \frac{1}{q} \left[r \left(1 - \frac{x_2^*}{1-s} \right) + \frac{\sigma}{x_2^*} \left(\frac{x_1^*}{s} - \frac{x_2^*}{1-s} \right) \right],$$

while the associated profit is

$$J^* = \frac{1}{q} \left[r \left(1 - \frac{x_2^*}{1-s} \right) + \frac{\sigma}{x_2^*} \left(\frac{x_1^*}{s} - \frac{x_2^*}{1-s} \right) \right] \frac{pqx_2^* - c}{\delta}.$$

Our job is now to reach the optimal solution optimally from the initial state $(x_1(0), x_2(0))$. This can be achieved by applying a ‘‘bang–bang’’ control (Pontryagin et al., 1962) to the system, as follows. Define

$$\tilde{E}(t) = \begin{cases} E_{\max} & \text{for } S(t) > 0, \\ E_{\min} & \text{for } S(t) < 0, \end{cases}$$

where $S(t) = pqx_2 - c - \lambda_2 qx_2$. Moreover, let T , be the time at which the path $(x_1(t), x_2(t))$, which is generated via the ‘‘bang–bang’’ control $E(t) = \tilde{E}(t)$, reaches the state (x_1^*, x_2^*) . Then, the optimal control policy is

$$E(t) = \begin{cases} \tilde{E} & \text{for } 0 \leq t \leq T, \\ E^* & \text{for } t > T \end{cases}$$

and the optimal path is given by the trajectory generated by the above optimal control. In view of the global stability property of the interior equilibrium of system (3), we can also reach the singular optimal solution through a suboptimal path by choosing the control policy $E(t)$ to be equal to E^* for all t . The advantage of choosing the optimal path is that it leads to the optimal singular solution more rapidly than does the suboptimal path.

7. Numerical simulations

We now illustrate the case of optimal harvesting at equilibrium numerically by using MATLAB. We specify $r = 0.5$, $s = 0.4$, $c = 0.4$, $q = 0.1$, $\delta = 0.005$, $\sigma = 0.4$, and

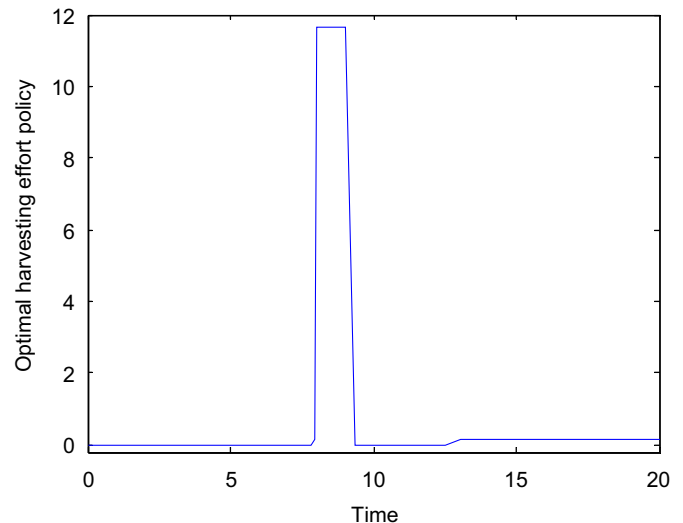


Fig. 8. Optimal harvesting effort policy.

$p = 7$. In addition, we set $E_{\min} = 0$ and $E_{\max} = 11.67$. For the above values of the parameters, we obtain an optimal singular effort of 0.17, an optimal singular stock of 0.39, 0.58, and an optimal singular revenue of 0.34.

The optimal harvest policy is presented in Fig. 8, and the associated optimal and suboptimal paths are presented in Figs. 9 and 10. From Figs. 9 and 10, it is clear that the

optimal and suboptimal paths approach their respective singular optimal solutions in both zones.

The details of the optimal and suboptimal paths presented in Figs. 9 and 10 are tabulated in Table 1. From this information, it is clear that the suboptimal paths take longer to reach the optimal singular solution than do the optimal paths.

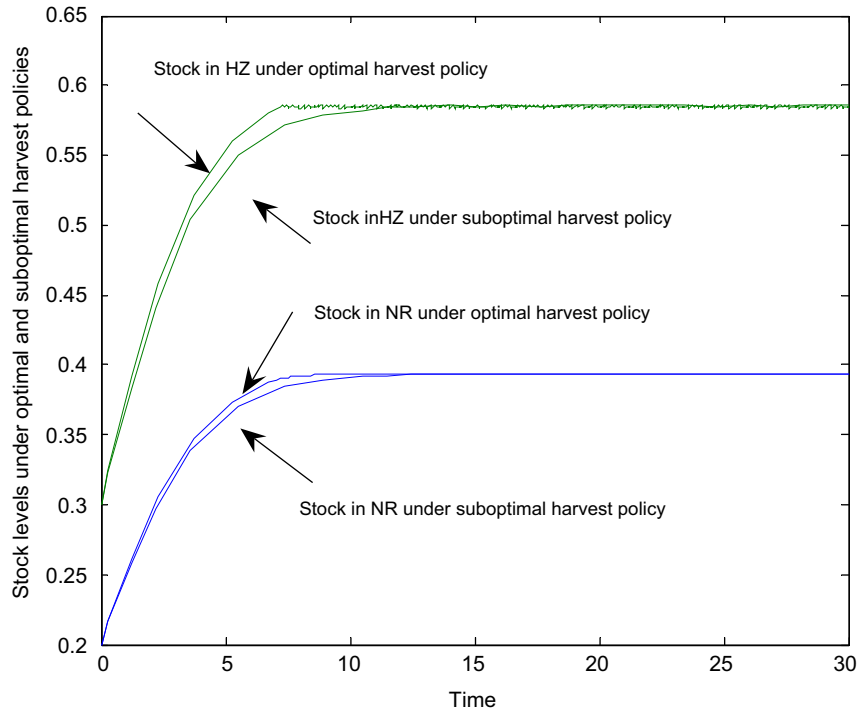


Fig. 9. This figure illustrates the optimal and suboptimal approach paths initiated at (0.2, 0.3).

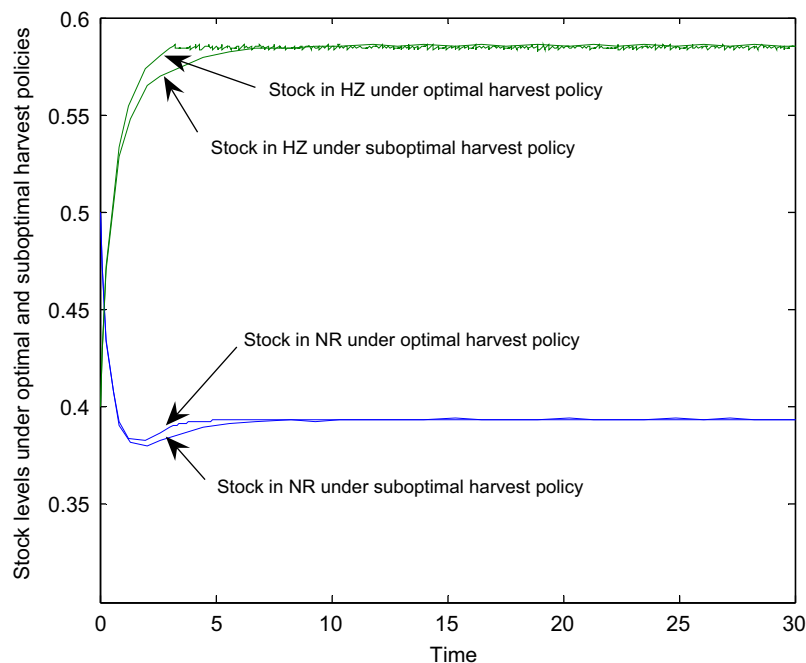


Fig. 10. This figure illustrates the optimal and suboptimal approach path initiated at (0.5, 0.4).

Table 1
Time taken by different approach paths

	Coordinates		Path type	Time taken to reach the end point
	Initial point	End point		
Fig. 9	(0.2, 0.3)	(0.17, 0.58)	Optimal	6.77 units
	(0.2, 0.3)	(0.17, 0.58)	Suboptimal	10.54 units
Fig. 10	(0.5, 0.4)	(0.17, 0.58)	Optimal	2.6 units
	(0.5, 0.4)	(0.17, 0.58)	Suboptimal	5.6 units

8. Concluding remarks

In this paper, we have discussed the impacts of MPAs from both economic and biological perspectives. The results of the analysis indicate that MPAs are a practical means of managing resource populations and are, therefore, beneficial for conserving the ecological environment and resource populations, despite the fact that extinction cannot be prevented in all cases. The establishment of MPAs could be used to maintain a high fish biomass in marine habitats. We have also proved that, when it exists, the interior equilibrium point is globally asymptotically stable. Ecological managers may find it desirable to achieve a unique positive equilibrium that is globally asymptotically stable, and they may use this goal in creating harvesting guidelines and developing the ecosystem in a sustainable manner.

We have also discussed the dynamic optimization of the harvest policy by taking $E(t)$, the harvest effort, as a dynamic variable. The optimal solution in the equilibrium case has also been discussed. Moreover, the biological and bioeconomic interpretations of the results associated with the optimal solutions have been explained.

The simulations (illustrated in Figs. 6 and 7) indicated that a larger reserve always increases resilience; however, a larger reserve does not necessarily increase the harvest. This result is obtained because, although a larger reserve increases spillovers to the fishery, it also reduces the proportion of the population that is subject to harvesting. We have also observed that the optimal and suboptimal paths approach their respective singular optimal solutions in both zones (see Figs. 9 and 10). From Table 1, it is clear that the suboptimal paths take a longer time to reach the optimal singular solution than do the optimal paths.

The methods and results provided in this paper do not constitute a comprehensive analysis. Nevertheless, we believe that our study is an important contribution toward the protection of fisheries, as it approximately characterizes the key level of harvesting.

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Appendix

To prove that the system is globally stable, we prove that the system does not tend to a limit cycle. Using the Bendixson–Dulac criterion (Strogatz, 1994), we can prove that the system does not have a limit cycle in the phase space. If such a closed trajectory C exists, then

$$\oint_C \left(\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \cdot \bar{n} \right) dl = 0$$
, where \bar{n} is the outward normal on C . The dot product must equal zero, because the trajectory follows C . Green's theorem yields $\iint_S \nabla \cdot \left(g(x_1, x_2) \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \right)$

$$= \oint_C g(x_1, x_2) \left(\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \cdot \bar{n} \right) dl$$
, where S is the surface enclosed by C .

Therefore, if we can find a function $g(x_1, x_2)$ for which the sign of the integrand is always positive or always negative over at least S , then the surface integral must not be equal to zero. Consequently, this means that C is not a trajectory. Taking $g(x_1, x_2) = 1/x_1x_2$, we have that

$$\begin{aligned} & \nabla \cdot \left(g(x_1, x_2) \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \right) \\ &= -\frac{1}{x_1} \left[\frac{r}{1-s} + \frac{\sigma x_1}{(1-s)x_2^2} \right] - \frac{1}{x_2} \left[\frac{r}{s} + \frac{\sigma x_2}{sx_1^2} \right] < 0. \end{aligned}$$

This implies that the system has no limit cycle. Hence, the interior equilibrium is globally asymptotically stable.

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