An extended cohort analysis: incorporating the effect of seasonal catches

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Abstract: A virtual population analysis relates changes in numbers at age of fish to commercial catch, fishing mortality, and natural mortality. We reformulate the standard virtual population analysis and corresponding catch equation in a manner that incorporates any seasonal pattern of catches, which may represent continuous fisheries or pulse fisheries. An advantage of this treatment is that it leads to a virtual population analysis formulated as cohort analysis, with $e^{M/2}$ replaced with a factor representing (and easily calculable from) the particular seasonal pattern of catches. The method presented here circumvents the standard assumption (leading to the Baranov catch equation) that fishing mortality is constant throughout the year.

Résumé : Une analyse de la population virtuelle établit un rapport entre les changements dans l'effectif selon l'âge du poisson et les prises commerciales, la mortalité par pêche et la mortalité naturelle. Nous reformulons l'analyse de la population virtuelle standard et l'équation de capture correspondante d'une façon qui intègre les profils saisonniers des captures, qui peuvent représenter des pêches en continu ou à caractère pulsatoire. L'avantage de cette méthode est qu'elle donne une analyse de la population virtuelle formulée comme une analyse de cohorte, $e^{M/2}$ étant remplacé par un facteur représentant (après un calcul facile) le profil saisonnier particulier des captures. La méthode présentée ici contourne l'hypothèse standard (qui mène à l'équation de capture de Baranov) selon laquelle la mortalité par pêche serait constante tout au long de l'année.
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Introduction

Most of the world's major fisheries are managed using statistical catch-at-age models that relate commercial catch-at-age to indices of abundance such as research surveys or commercial catch per unit effort indices (Hilborn and Walters 1992). These models describe the relationship between population abundance, fishing mortality, natural mortality, and the commercial catch. The most common assumption made is that fishing mortality is constant throughout the year, which leads to the Baranov catch equation. The Baranov catch equation implies a nonlinear equation for virtual population analysis (VPA); that is, the numbers of fish in a cohort in year i are not linearly related to either the catch or the numbers in year $i + 1$. The VPA equation may be solved iteratively or approximated using a variety of approaches (Pope 1972; Sims 1982*a*; MacCall 1986; Allen and Hearn 1989; Evans 1989). Of these approaches, the one that is almost always used in practice is Pope's approximation (described below), which allows a linear VPA. Pope's formulation is exact for a pulse fishery occurring at midyear, although it is often regarded as an approximation to the Baranov-based approach (Hilborn and Walters 1992).

The methods listed above generally assume that the Baranov catch equation is exact (or occasionally that Pope's formula is exact). In fact few fisheries will conform to either the Baranov pattern of constant fishing mortality throughout the year (MacCall 1986; Evans 1994) or the Pope pattern (a pulse fishery at midyear). Here, we present a method that explicitly allows seasonal variations in catch rates (catch per unit time) and that may represent continuous fisheries or pulse fisheries. This technique provides the benefit of a VPA equation that takes the form of Pope's (1972) cohort analysis, requiring only the replacement of $e^{M/2}$ with a factor representing (and that is easily calculable from) the seasonal pattern of catches for the stock in question. Our approach further allows the derivation of a generalized catch equation, which is not limited to a Baranov type fishery. Our results are easily implemented in any of the standard approaches to the analysis of catch-at-age data.

Analysis

We will proceed from the following equation, governing the evolution of numbers in a cohort (N) through time:

$$\frac{dN}{dt} = -mN - C' \qquad [1]$$

where m is the instantaneous natural mortality rate (having units of time to the minus one) and $C' \equiv dC/dt$ (C represents

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catch) is the number of fish being caught, at an instant, per unit time; that is, dC/dt is the instantaneous catch rate. The instantaneous natural mortality rate is related to the annual mortality rate M through $M = m \times (1\ \text{year})$ when M is constant.

Assuming only that m is constant, a general solution to eq. 1 (see, e.g., Ritger and Rose 1968, section 2-3) can be obtained:

$$[2] \quad N(t_1) - e^{-mt_1} N(0) = -e^{-mt_1}\int_0^{t_1} e^{mt}\, C'\mathrm{d}t$$

where $N(0)$ is the initial number in the cohort and $N(t_1)$ is the number at $t = t_1$.

As catch data are generally available on a per annum basis, we will now recast eq. 2 to reflect our interest in annual changes. Now, $t = 0$ in eq. 2 will be the beginning of a given year, indexed by i, and $t = t_1$ will be the end of the year. Thus, $N(0)$ becomes N_i and $N(t_1)$ becomes N_{i+1}. Also, since t_1 = 1 year, we have $mt_1 = M$. In all forthcoming equations, t will refer to time within the span of a year and will be expressed as a fraction in the range 0–1. Equation 2 may now be written as

$$[3] \quad N_{i+1} - e^{-M}N_i = -e^{-M}\int_0^1 e^{Mt}\, C_i'(t)\,\mathrm{d}t$$

We have also made the catch rate year specific. Equation 3 includes the limit in which the catch is taken in a series of pulse type fisheries, as envisioned by Sims (1982*b*) and MacCall (1986). Sims (1982*b*) applied Pope's formula to each of a number of subintervals of a year, and thus his eq. 3 will be exact if the fishery consists of series of pulse catches at the middle of each subinterval. Similarly, MacCall (1986), in his eq. 15, derives a population equation that is exact for a series of pulse catches at known times.

By writing $\tilde{C}_i = e^{-M}\int_0^1 e^{Mt}\, C_i'(t)\,\mathrm{d}t$, we can express eq. 3 as

$$[4] \quad N_{i+1} - e^{-M}N_i = -\tilde{C}_i$$

where $\tilde{C}_i$ could be termed the adjusted catch, in that it represents a weighted integral over time of the catch rate. If catch rate data are available, perhaps on a monthly basis, for the fishery of interest, the integral in eq. 3 may be replaced with the appropriate sum over months. Equation 4 provides for a simple VPA; once the $\tilde{C}_i$ values are calculated, N_i is easily computed from N_{i+1}. In fact the VPA formulated has the advantage that no fishing mortality term appears, which would necessitate (as in the standard VPA) a new approximation to linearize the equation or an iterative solution. Equation 4 as it stands is complete, and the analysis could be terminated here. Although eq. 4 is sufficient for a VPA analysis, we have not dealt with the issue of how significantly the adjusted catch $\tilde{C}_i$ departs from the actual catch C_i that is given by $\int_0^1 C_i'\,\mathrm{d}t$. This will be taken up immediately.

To compare C_i and $\tilde{C}_i$ we introduce the ratio

$$[5] \quad \gamma_i(M) = \frac{\tilde{C}_i}{C_i} = \frac{e^{-M}\int_0^1 e^{Mt}\, C_i'(t)\,\mathrm{d}t}{C_i}$$

This ratio gauges the extent to which the seasonal pattern in catch rate affects the population analysis. We will be particularly interested in cases for which $\gamma_i(M)$ can be treated as a constant (negligible interannual variations). In the material to be presented we will write $\gamma_i(M)$ as $\gamma(M)$ in cases where the interannual variability can be ignored.

A constant $\gamma_i(M)$ will result if the seasonal catch rate pattern does not change from year to year. The factor $\gamma_i(M)$ will be constant if $C_i'(t)$ can be written as $C_i'(t) = A_i\, c'(t)$, a year effect, A_i, multiplied by a catch rate profile, $c'(t)$, where $c'(t)$ describes the changes in catch rate within a year. For example, we could specify $c'(t) = \sin(\pi t)$, a function that describes a fishery peaking in midyear.

We can now write eq. 3 as

$$[6] \quad N_{i+1} - e^{-M}N_i = -\gamma_i(M)C_i$$

Below we will establish bounds for $\gamma_i(M)$ and we will describe situations under which $\gamma_i(M)$ may be treated as a constant.

Extended cohort analysis

Equation 6 can be rewritten as

$$[7] \quad N_i = e^{M}N_{i+1} + \gamma_i(M)e^{M}C_i$$

which may be recognized as a close analog of cohort analysis (Pope 1972). In fact, by specifying a pulse fishery at midyear, eq. 5 immediately yields $\gamma_i(M) = \gamma(M) = e^{-M/2}$ and, after substitution into eq. 7, one recovers Pope's (1972) result. The so-called discrete fishery (Hilborn and Walters 1992; section 10.2) consists of a pulse fishery at the end of the year; one can immediately calculate $\gamma(M) = 1$, which yields the correct VPA equation when substituted into eq. 7. In general, a pulse fishery at time α $(0 \le \alpha \le 1)$ gives $\gamma(M) = e^{-M(1-\alpha)}$ with the corresponding VPA equation being

$$[8] \quad N_i = e^{M}N_{i+1} + e^{-M\alpha}C_i$$

which is the natural generalization of cohort analysis for pulse fisheries.

These results for the pulse fisheries have a straightforward interpretation: in eq. 6 we fully discount N_i by the factor e^{-M}; however, if the catch is removed immediately at the beginning of the year, we actually should discount the numbers given by $N_i - C_i$, and hence the appearance of the term $e^{-M}C_i$ in eq. 6, for this case. If the catch is taken at the end of the year, then full discounting of N_i is appropriate and no product of natural mortality and catch term arises. In view of this, we might term $\gamma(M)$ the mortality correction factor.

The limits for $\gamma_i(M)$ are defined by the two extreme cases of a pulse fishery at the beginning of the year and a pulse fishery at the end of the year. For a pulse fishery at the end of the year, no correction is necessary for the application of natural mortality and $\gamma_i(M) = 1$. Pulse fishing at the beginning of the year implies that we have overdiscounted N_i by the maximum amount possible, specifically, $C_i e^{-M}$ (in other words, $\gamma_i(M) = e^{-M}$). Thus, with complete generality we can state

$$[9] \quad e^{-M} \le \gamma_i(M) \le 1$$

These considerations explain the success of Pope's (1972) cohort analysis: by in effect adopting the compromise value of $\gamma_i(M) = \gamma(M) = e^{-M/2}$, Pope's treatment can never be in error by more than $e^{-M/2} - e^{-M}$ or $1 - e^{-M/2}$. If M is not too large then both these expressions are approximately equal to $M/2$, so that for a typical M of 0.2 the error can never be worse than about 10%.

Treating γ as a constant

In this subsection we address the issue of when $\gamma_i(M)$ can be treated as a constant, i.e., when its interannual variability can be ignored. The direct procedure is to calculate $\gamma_i(M)$ numerically for each year, say from monthly catch rate data (catch per month):

$$[10]\quad \gamma_i(M) = \frac{e^{-M}\sum_{n=1}^{n=12} e^{((n-1/2)/12)M} C_i'(n)}{\sum_{n=1}^{n=12} C_i'(n)}$$

where we have assumed that the catch rate data represent the center of each month, indexed by n (n = 1 to 12). Of course, more sophisticated numerical approximations to the integral in eq. 5, such as Simpson's rule, can be employed, if desirable. The calculated values of $\gamma_i(M)$ can then be inspected to determine if the interannual variations are small enough to be neglected. However, some simple analysis, presented below, shows that numerical treatment is unnecessary in many cases.

For an analytic treatment, consider first the example of a seasonally limited fishery with reasonably constant catch rates. This might occur when a fisheries operations manager matches catches to plant capacity, say maintaining the catch rate at a specified number of tons per week. Consider the case where the fishery is centered on midyear, for comparison to cohort analysis; specifically, the catch is taken at a constant rate for a period from $0.5 - \alpha/2$ to $0.5 + \alpha/2$. We find

$$[11]\quad \gamma(M) = e^{-M/2}\,\frac{e^{M\alpha/2} - e^{-M\alpha/2}}{M\alpha}$$

Clearly as α shrinks to zero the cohort analysis result of $\gamma(M) = e^{-M/2}$ is recovered. However, even in the extreme case of $\alpha = 1$, agreement with the cohort analysis result is very good. From eq. 11, $\gamma(0.2) = 0.906$ and $\gamma(0.5) = 0.787$, while from the cohort analysis, $\gamma(0.2) = 0.905$ and $\gamma(0.5) = 0.779$. For a sinusoidal catch rate profile, $c'(t) = \sin(\pi t)$, $\gamma(M) = (1/2)(1 + e^{-M})((M/\pi)^2 + 1)^{-1}$, we have $\gamma(0.2) = 0.906$ and $\gamma(0.5) = 0.783$. It is evident that cohort analysis, stipulating a pulse fishery at midyear, is an excellent approximation to the two continuous fisheries treated, constant catch throughout the year and the sinusoidal catch rate profile.This is a consequence of the fact that these latter two fisheries are symmetric about their centroids, so that $\gamma(M)$ is extremely insensitive to the shape of the catch rate profile, and its year-to-year constancy is assured. California mackerel (Fig. 2 of MacCall 1986) provides a possible example in that the monthly catches are approximately symmetrical about midyear.

There will be cases where the catch rate profile varies appreciably from year to year; for example, fishing mortality could be constant within a year (a Baranov type fishery) but vary among years, in which case the shape of the catch rate curve changes interannually. In many such examples it will be possible to exploit seasonal limitations on the fishery to justifiably invoke a constant $\gamma(M)$. If the fishery is confined to the interval α_1 to α_2 ($\alpha_2 \ge \alpha_1$, $0 \le \alpha_1, \alpha_2 \le 1$) then, using the same arguments used to justify eq. 9, we can write

$$[12]\quad e^{-M\alpha_2} \le \gamma_i(M) \le e^{-M\alpha_1}$$

which is, of course, a direct analog of eq. 9. The natural choice for $\gamma_i(M)$ is simply, in the spirit of Pope's cohort analysis, $\gamma_i(M) = \gamma(M) = e^{-M(\alpha_1 + \alpha_2)/2}$. If the arguments of the exponentials are not large, the maximum possible error resulting from this choice of $\gamma_i(M)$ will be approximately $M(\alpha_2 - \alpha_1)/2$. For a 3-month fishery ($\alpha_2 - \alpha_1 = 0.25$), with $M = 0.2$ the maximum possible error is about 2.5%. Even for $M = 0.5$ the maximum possible error is about 6%. Bear in mind that these errors are extremes that would only be encountered if the fishery fluctuated between a pulse fishery at the beginning of the 3-month interval and one at the end of the 3-month interval. Thus, seasonal limitation of the fishery will often justify treatment of $\gamma(M)$ as being constant from year to year (despite changes in the catch rate profile).

A related problem will arise if it is suspected that the centroids of the catch versus time (within a year) curves change with age group. The significance of the shift can be assessed by applying an error analysis similar to that given in the previous paragraph. One can easily show that $\Delta\gamma_i(M)/\gamma_i(M) \approx M\Delta\alpha$, where $\Delta\alpha$ is the centroid shift and $\Delta\gamma_i(M)$ is the corresponding change in $\gamma_i(M)$. Thus, a 3-month difference in the centroids of the catches of two age groups ($\Delta\alpha = 0.25$) would result in a 2.5% change in $\gamma_i(M)$ (for $M = 0.2$).

Finally, we have applied eq. 10 to two real fisheries, as described in the next two paragraphs.

We calculated $\gamma_i(M)$ for northern cod, i.e., cod off Labrador and the northern Grand Banks. This resource has traditionally been prosecuted primarily during the summer and was one of the stocks for which Pope developed his approximation. We have calculated $\gamma_i(M)$ (Fig. 1), for $M = 0.2$ (the accepted assessment value), for each year from 1954 to 1991 using data compiled and described by Hutchings and Myers (1995). The seasonal patterns for the catches for the years of maximum and minimum $\gamma_i(M)$ are shown in Fig. 1B. Pope's approximation (broken line) is usually correct within 3% for this fishery. The trend in the $\gamma_i(M)$ over time is caused by interannual shifts in the timing of the offshore fishery. In the early 1950s there was virtually no spring fishery because of sea ice. Trawlers that could operate in ice were introduced in the late 1950s and the largest catches were taken in the spring on prespawning concentrations (Hutchings and Myers 1995). Despite the changes in seasonality, the variations in $\gamma_i(M)$ are generally weak (for reference note that northern cod catches changed by more than a factor of four in the plotted period). Although the northern cod fishery is far from being a pulse fishery, it is evident that the Pope (1972) formulation excellently approximates the true $\gamma_i(M)$.

In Fig. 2 we present the $\gamma_i(M = 0.2)$ series, and seasonal catch patterns for the years of maximum and minimum γ_i, for herring of the east and south coasts of Newfoundland. Herring catches are generally bimodal with peaks in the spring and fall, and there is a secular shift evident, from a dominant peak in spring, for the early segment of the record, to a dominant peak in fall, for the latter portion of the record. In the late 1980s the discrepancy between the true $\gamma_i(M)$ and Pope's approximation becomes greater than 5%, but it is still notable that Pope's approximation remains quite reliable even for this fishery that does not conform to the Pope pattern of peaking at midyear.

To summarize the material presented thus far, in this section, we note that our examination of the behavior of $\gamma_i(M)$ has shown that its interannual variability will in many

Fig. 1. (A) The factor $\gamma_i(M = 0.2)$ calculated from eq. 10 for northern cod, i.e., cod off Labrador and the northern Grand Banks. The broken line is Pope's approximation. (B) Monthly catches expressed as percentages of the total for the years of maximum and minimum $\gamma_i(M)$.

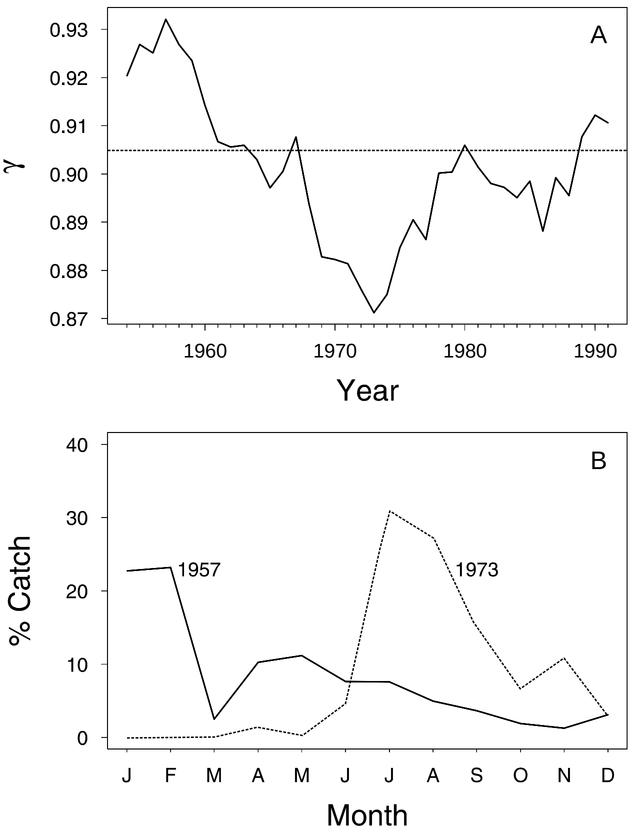
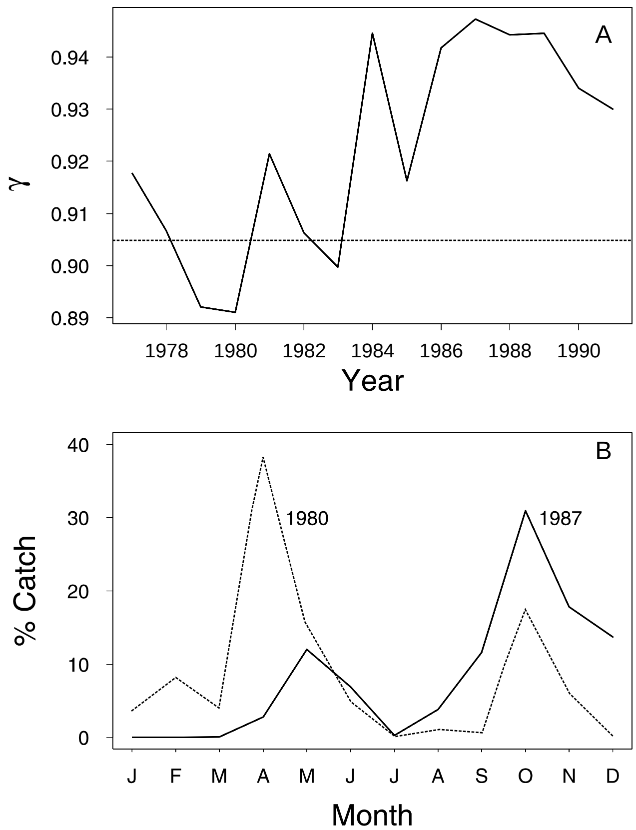


Fig. 2. (A) The factor $\gamma_i(M = 0.2)$ calculated from eq. 10 for south and east coast Newfoundland herring. The broken line is Pope's approximation. (B) Monthly catches expressed as percentages of the total for the years of maximum and minimum $\gamma_i(M)$.



circumstances be much less than that of catches, suggesting that catches can be entered in the population model with a constant $\gamma(M)$. (Equivalently, the interannual variability in the adjusted catch, $\tilde{C}_i$, will be almost entirely due to changes in catch.) By formulating our population model, eq. 6 or eq. 7, with the factor $\gamma_i(M)$ we have emphasized the continuity and compatibility with Pope's (1972) cohort analysis.

The catch equation

A general catch equation, not specific to any particular seasonal fishing pattern, may be obtained from eq. 6. We require the following equation:

$$[13] \quad N_{i+1} = N_i e^{-(F_i+M)}$$

where F_i is the fishing mortality (which is defined by this equation). Equations 6 and 13 may be used to show that

$$[14] \quad \gamma_i(M)C_i = \tilde{C}_i = N_i(e^{-M} - e^{-(F_i+M)}) = (1 - e^{-(F_i+M)})N_i - (1 - e^{-M})N_i$$

Equation 14 may be interpreted as follows: the adjusted catch is equal to the numbers perishing from fishing plus natural mortality less the numbers that would perish from natural mortality if the entire catch were taken at the end of the year. Note that the catch is bounded by the limits $e^{-M} N_i(1 - e^{-Fi})$ and $N_i(1 - e^{-Fi})$, a result derived earlier by Mesnil (1980).

The standard, or Baranov, catch equation is given by

$$C_i = \frac{F_i}{F_i + M}(1 - e^{-(F_i+M)})N_i$$

and it may be shown to be a special case of eq. 14. If we specify constant fishing mortality throughout the year, $C_i(t) = A_i \exp(-(F_i + M)t)$, then substitution of this form of $C_i(t)$ into eq. 5 yields

$$[16] \quad \gamma_i(M) = \frac{F_i + M}{F_i} \frac{e^{-M} - e^{-(F_i+M)}}{1 - e^{-(F_i+M)}}$$

and substitution of this $\gamma_i(M)$ into eq. 14 gives the Baranov catch equation, as it must.

Discussion

Many previous studies have taken the Baranov catch equation to be exact, and considerable effort has been expended to preserve this exactness in the VPA, by finding accurate, rapidly converging iterative solutions or accurate linear approximations to the nonlinear VPA resulting from the Baranov catch equation. Once it is acknowledged that the Baranov catch equation is, in general, an approximation (because catch rates may be very seasonal), considerable simplification is possible. The approach encapsulated in eq. 6 and eq. 7 permits the incorporation of realistic catch rate profiles while also providing the benefit of a VPA that is as simple as, and closely analogous to, cohort analysis. The significance of interannual variations in $\gamma_i(M)$ can be investigated through application of the bounds in eq. 12, or through direct calculation (using eq. 10) when detailed catch information is available. By applying eq. 10 to the northern cod fishery we were able to demonstrate that Pope's approximation is accurate to within about 3% for this example. In cases where the interannual variability of $\gamma_i(M)$ is not negligible, one still acquires, with our approach, the advantage of a VPA resembling cohort analysis (N_i is immediately calculable from N_{i+1} and C_i) once the $\gamma_i(M)$ series is calculated.

We should emphasize that the accuracy of Pope's, or any other, approximation for a particular fishery should be established by estimating the $\gamma_i(M)$ rather than by comparing the approximation with the Baranov result. In fact, we would recommend calculating the $\gamma_i(M)$ series whenever sufficient catch data exist. If monthly catch data are not available, $\gamma_i(M)$ can still be estimated from approximate knowledge of the timing of the fishery (see discussion near eq. 12) and the adequacy of the estimate of $\gamma_i(M)$ can be evaluated using eq. 12.

A VPA formulation taking the form of a generalized cohort analysis, such as eq. 7, is a considerable advantage to methods that employ Kalman filters and state space models, which have relied upon the linearity to incorporate estimation error in the catch-at-age (Collie and Sissenwine 1983; Mendelssohn 1988; Gudmundsson 1994). Linearity may be achieved by assuming that the fish are caught in a pulse at the beginning of the year (Collie and Sissenwine 1983; Mendelssohn 1988; Walters and Punt 1994). This is equivalent to assuming that $\gamma_i(M) = e^{-M}$. Our method can be used to provide the $\gamma_i(M)$ appropriate to the actual catch profile, and in many cases will justify its treatment as a constant. The same linearity means that analytical derivatives can be easily estimated in models with complex error structure (Walters and Punt 1994; Myers and Cadigan 1995) or more complex models that include size structure (Schnute et al. 1989). In the catch-at-age model most often used by the International Council for the Exploration of the Sea, called extended survivors (J.G. Shepherd, MAFF Fisheries Laboratory, Suffolk NR33 OHT, England, unpublished data), Pope's (1972) approximation is key to the estimation algorithm. All of these methods can be easily generalized by using our results.

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